Canonical Forms for Linear Systems

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ABSTRACT

This paper considers canonical forms for the similarity action of $\mathrm{Gl}(n)$ on $\Sigma_{n,m} = \{(A,B) \in \mathbb{C}^{n \cdot n} \times \mathbb{C}^{n \cdot m}\}$:

$$\begin{aligned} \mathrm{Gl}(n) \times \Sigma_{n,m} &\to \Sigma_{n,m}, \\ (H,(A,B)) &\mapsto (HAH^{-1}, HB) \end{aligned} \tag{$*$}$$

Those canonical forms are obtained as an application of a more general method to select canonical elements M_c in the orbits $\mathcal{O}_M$ of a matrix group G acting on a set of matrices $\mathfrak{M} \subset \mathbb{C}^{l \cdot p}$. We define a total order $(\prec)$ on $\mathbb{C}^{l \cdot p}$, different from the lexicographic order $\overset{l}{\prec}$ $[0 \overset{l}{\prec} x \Leftrightarrow x < 0$, but $0 \prec x \Leftrightarrow x \neq 0$ for $x \in \mathbb{R}]$ and consider normalized $\mathcal{O}_M$-elements with a minimal number of parameters:

$$\min_{\prec} \{\hat{M} \in \mathcal{O}_M : \hat{M} \text{ normalized}\}.$$

It is shown that the row and column echelon forms, the Jordan canonical form, and "nice" control canonical forms for reachable (A,B)-pairs have a homogeneous interpretation as such $(\prec)$-minimal orbit elements. Moreover new canonical forms for the general action $(\prec)$ are determined via this method.

1. INTRODUCTION

Time-invariant linear systems in the state-space description,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

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or

$$x(k+1) = Ax(k) + Bu(k), \tag{1.2}$$

are usually identified with elements (A, B) of the affine spaces

$$\Sigma_{n,m} := \mathfrak{M}_{n,n} \times \mathfrak{M}_{n,m}, \tag{1.3}$$

where $\mathfrak{M}_{n,m}$ denotes the space of all $n \times m$ matrices with entries in the complex field[1] $\mathbb{C}$, n and m are in $\mathbb{N}$. Pairs (A, B) belonging to the same orbit $\mathcal{S}_{(A,B)}$ of the natural $\mathrm{Gl}(n)$-action

$$\begin{aligned} \mathrm{Gl}(n) \times \Sigma_{n,m} &\to \Sigma_{n,m} \\ (H,(A,B)) &\mapsto (HAH^{-1}, HB) \end{aligned} \tag{1.4}$$

are called *similar*.

A *canonical form* on $\Sigma_{n,m}$ is a map $c: \Sigma_{n,m} \to \Sigma_{n,m}$ with the properties

$$c(A,B) \in \mathcal{S}_{(A,B)}, \tag{1.5}$$

$$c(A,B) = c(\hat{A},\hat{B}) \quad \Leftrightarrow \quad \mathcal{S}_{(A,B)} = \mathcal{S}_{(\hat{A},\hat{B})}. \tag{1.6}$$

These canonical forms are investigated in the following special cases:

(i) $\dot{x}(t) = Ax(t)$, $B = 0$. For completely uncontrollable (free) systems (1.4) reduces to the similarity action on square matrices

$$\begin{aligned} \mathrm{Gl}(n) \times \mathfrak{M}_{n,n} &\to \mathfrak{M}_{n,n}, \\ (H,A) &\mapsto HAH^{-1}. \end{aligned} \tag{1.7}$$

The orbits $\mathcal{S}_A$ are parametrized by the *Jordan canonical form*.

[1] Throughout this paper we restrict ourselves to the field $\mathbb{C}$ of complex numbers. The main reason is to avoid the real Jordan canonical form in later sections. The paper can be easily extended to $\mathbb{R}$ or to ordered fields.

(ii) $\dot{x}(t) = Bu(t)$, $A = 0$. For integrators (1.4) coincides with row operations

$$\begin{gathered} \mathrm{Gl}(n) \times \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}, \\ (H, B) \mapsto HB. \end{gathered} \tag{1.8}$$

The orbits $\mathcal{R}_B$ contain canonical *row echelon matrices* (see Section 2).

(iii) $\dot{x}(t) = Ax(t) + Bu(t)$ *with* $\operatorname{rank}[B\ AB\ \cdots\ A^{n-1}B] = n$. Various canonical forms have been proposed in the literature (e.g. Popov [10], Mayne [9], Weinert and Anton [14], Denham [3], Rissanen [12]) for the similarity action on the space of *reachable* (A, B)*-pairs*:

$$\Sigma^r_{n,m} := \{(A, B) \in \Sigma_{n,m} : \operatorname{rank}[B\ AB\ \cdots\ A^{n-1}B] = n\}. \tag{1.9}$$

In particular it was observed that most of the so-called standard or canonical forms presented in earlier publications (Brunovsky [1], Luenberger [8], Rosenbrock [13]) do not satisfy the requirements of the above definition (cf. [10, 3]). Hazewinkel and Kalman [5] showed that there does not exist a *continuous* global canonical form for $\Sigma^r_{n,m}$. This result also destroys any hope of finding those canonical forms for $\Sigma_{n,m}$ itself. However, this should not prejudice the investigation of global canonical forms for $\Sigma_{n,m}$. As the Jordan canonical form for single matrices illustrates, global canonical forms may be very useful without being continuous.

So far as we know, there is only one paper treating the nonreachable case. Byrnes and Gauger [2] do so for *scalar systems* ($m = 1$) a canonical form $({}^J\!A, {}^J b)$ with ${}^J\!A$ in Jordan canonical form and ${}^J b \in \{0,1\}^n$. However, their result contains an error with respect to the number of 1-entries in ${}^J b$ (see Section 5). Moreover, there is no obvious way to generalize their approach to *multivariable systems* ($m > 1$). In this paper we establish such a generalization as an application of a more general method to derive canonical forms for matrix orbits.

Let $\mathcal{G} \times \mathfrak{M} \to \mathfrak{M}$ be any action of a group $\mathcal{G}$ of matrices on a set $\mathfrak{M}$ of $n \times m$ matrices. A general principle for singling out canonical elements M_c in the orbits $\mathcal{O}_M$, $M \in \mathfrak{M}$, of such an action is to reduce the number of parameters of the orbit elements as much as possible. This can be achieved systematically by applying the following total order ($\prec$): Let $x, y \in \mathbb{R}^k$, $k \in \mathbb{N}$. Define the *leading index* and the *leading coordinate* of x by

$$l(x) := \begin{cases} \max\{i; x_i \neq 0\} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{1.10}$$

$$x_l := \begin{cases} x_{l(x)} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases} \tag{1.11}$$

The order ($\prec$) is defined as follows:

$$x \prec y \quad :\Leftrightarrow \quad \begin{cases} x_{l(x-y)} = 0 \quad \text{or} \\ x_{l(x-y)} \neq 0 \ \wedge \ x_{l(x-y)} < y_{l(x-y)}. \end{cases} \tag{1.12}$$

EXAMPLE 1.1.

$$\begin{pmatrix} 0 \\ 3 \\ 4 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} \quad \text{because} \quad x_{l(x-y)} = x_4 = 0,$$

$$\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 3 \\ -2 \\ 1 \end{pmatrix} \quad \text{because} \quad x_{l(x-y)} = x_3 = 0,$$

$$\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix} \quad \text{because} \quad x_{l(x-y)} = 3 < y_{l(x-y)} = 4.$$

($\prec$) coincides with the lexicographic order ($\overset{l}{\prec}$) on $\mathbb{R}^N$ if we modify the usual order on $\mathbb{R}$ by

$$0 \prec x \quad \Leftrightarrow \quad x \in \mathbb{R}\backslash\{0\}$$

and

$$x \prec y \ \Leftrightarrow \ x < y \qquad \text{for} \quad x, y \in \mathbb{R}\backslash\{0\},$$

i.e., define 0 to be the smallest real number.

Identifying $a + bi \in \mathbb{C}$ with

$$\begin{pmatrix} b \\ a \end{pmatrix} \in \mathbb{R}^2,$$

we can apply the order ($\prec$) also to $\mathbb{C}^k$, $k \in \mathbb{N}$. In order to apply ($\prec$) to $n \times m$ matrices A we have to identify the matrices in some way with

nm-column vectors. Among all possibilities we select the following columnwise and rowwise procedures:[2]

$$A \underset{c}{\prec} \hat{A} \quad :\Leftrightarrow \quad \begin{pmatrix} a^m \\ \vdots \\ a^1 \end{pmatrix} \prec \begin{pmatrix} \hat{a}^m \\ \vdots \\ \hat{a}^1 \end{pmatrix}, \tag{1.13}$$

$$A \underset{r}{\prec} \hat{A} \quad :\Leftrightarrow \quad \begin{pmatrix} (a_m)^T \\ \vdots \\ (a_1)^T \end{pmatrix} \prec \begin{pmatrix} (\hat{a}_m)^T \\ \vdots \\ (\hat{a}_1)^T \end{pmatrix}, \tag{1.14}$$

respectively.

The simplest idea to obtain canonical elements for the above mentioned matrix orbits $\mathcal{O}_M$ is to consider

$$\min_{\underset{c}{\prec}} \{ \hat{M} \in \mathcal{O}_M \}.$$

But in general those elements do not exist. However, normalizing a sufficient number of parameters for the $\mathcal{O}_M$ elements we in general obtain uniquely determined ($\prec$)-minimal orbit elements.

To determine those elements we can proceed in most cases in two steps. Apply first the *screen function*

$$\omega : \mathbb{C}^k \to \{0,1\}^k, \qquad \omega(x)_i := \begin{cases} 0 & \text{if} \quad x_i = 0, \\ 1 & \text{if} \quad x_i \neq 0 \end{cases} \tag{1.16}$$

to the orbit $\mathcal{O}_M$, and determine the minimum

$$w^* = \min_{\underset{c}{\prec}} \{ \omega(\hat{M}) : \hat{M} \in \mathcal{O}_M \}.$$

Because ($\prec$) is a total order and $\{0,1\}^k$ is finite, this minimum exists. The

[2] Throughout this paper matrices are denoted by capital letters. m^j denotes the jth column, m_i the ith row, and m_i^j the (i, j)th entry of the matrix M.

associated subset $\Omega := \{\hat{M} \in \mathcal{O}_M : \omega(\hat{M}) = w^*\}$ contains in general many elements, but from the particular group action it is often clear how canonical elements in Ω can be maintained by normalizing certain entries of the Ω-elements.

Proceeding in this way, we derive in Sections 2, 3, and 4 $(\prec)$-characterizations of the *echelon forms*, the *nice control canonical forms* for $\Sigma^r_{n,m}$, and the *Jordan canonical form*. In Section 5 we drop the reachability assumption and derive Jordan canonical pairs $({}^JA, {}^JB)$ for $\Sigma_{n,m}/\mathrm{Gl}(n)$. The ${}^JB$-matrices are obtained as normalized $(\prec)$-minimal elements of the orbits $\mathcal{J}_{(A,B)}$ of the action $\mathrm{Stab}({}^JA) \times \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}$, $(H, B) \mapsto HB$. Finally in Section 6 we combine the nice control canonical forms with the Jordan canonical form and obtain composite canonical orbit elements of the form

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where (A_1, B_1) is a nice control canonical form for the reachable subsystem of (A, B), A_3 is in Jordan canonical form, and A_2 is somehow canonically fixed with a few parameters.

It should be mentioned that the emphasis of this paper lies on the unifying concept of the order $(\prec)$, i.e., to demonstrate that apparently different canonical forms can be given a homogeneous interpretation as normalized $(\prec)$-minimal elements. However, besides interpretations of known canonical forms in terms of $(\prec)$, also new canonical forms for the general $\mathrm{Gl}(n)$-action on $\Sigma_{n,m}$ are derived with the help of $(\prec)$.

2. ECHELON FORMS

Let $A \in \mathfrak{M}_{n,m}$, rank $A = r$. Row operations $A \mapsto LA$ and combinations with column operations $A \mapsto LAR$ can be interpreted as group actions

$$G \times \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}, \tag{2.1}$$

where G stands for $\mathrm{Gl}(m)$ or $\mathrm{Gl}(m) \times \mathrm{Gl}(n)$. It is well known that the associated orbits $\mathcal{R}_A$ and $\mathcal{CR}_A$ contain canonical *echelon matrices* ${}^RA$ and ${}^{CR}A$ of the form

$${}^RA = [A(\sigma_1), A(\sigma_2), \ldots, A(\sigma_r)] \tag{2.2}$$

with

$$A(\sigma_i)=\begin{array}{c}\overbrace{\hphantom{0\ \cdots\ 0\ \ 0}}^{\sigma_i}\\ \left[\begin{array}{cccc}0&\cdots&0&0\\ \vdots&&\vdots&\vdots\\ 0&\cdots&0&0\\ *&&*&1\\ *&\cdots&*&0\\ \vdots&&\vdots&\vdots\\ *&\cdots&*&0\end{array}\right]\end{array}\left.\begin{array}{c}\\ \\ \\ \\ \\ \\ \\ \end{array}\right\}r-i+1, \tag{2.3}$$

$$A(\sigma_r)=\begin{bmatrix}0&\cdots&0&0&0&\cdots&0\\ \vdots&&\vdots&\vdots&\vdots&&\vdots\\ 0&\cdots&0&0&0&\cdots&0\\ *&\cdots&*&1&0&\cdots&0\end{bmatrix}$$

and

$$\begin{aligned}\sigma_i &:=\gamma_i-\gamma_{i-1},\\ \gamma_0 &:=0,\end{aligned} \tag{2.4}$$

where

$$\operatorname{rank}\left[a^1,\dots,a^{\gamma_i}\right]=i, \tag{2.5}$$

$$\operatorname{rank}\left[a^1,\dots,a^{\gamma_i+1}\right]=i+1 \qquad \text{for} \quad i=0,\dots,r-1; \tag{2.6}$$

and

$${}^{CR}A=\left[\begin{array}{c|ccc}0_{r,m-r}&1&&\\ &&\ddots&\\ &&&1\\ \hline 0_{n-r,m-r}&&0_{n-r,r}&\end{array}\right]. \tag{2.7}$$

The ($\prec$)-interpretation of these forms is as follows:

PROPOSITION 2.1.1.

$$^{R}A = \min_{\underset{r}{\prec}}\{\hat{A}\in \mathfrak{R}_A : (\hat{a}_i)_l = 1 \textit{ for } i \in \underline{n}\}, \tag{2.8}$$

$$^{CR}A = \min_{\underset{c}{\prec}}\{\hat{A}\in \mathcal{C}\mathfrak{R}_A : (\hat{a}^i)_l = 1 \textit{ for } i \in \underline{m}\}. \tag{2.9}$$

Proof. The echelon forms ^{R}A and ^{CR}A fulfill the normalizability conditions $(^{R}a_i)_l = 1$, $(^{CR}a^i)_l = (^{CR}a_i)_l = 1$. The uniqueness of the echelon forms implies that any normalized $\hat{A}$ with $\hat{A} \underset{r}{\prec} {}^{R}A$ has at least one 0-entry in a position where ^{R}A has a leading 1 and the same fixed 0-block above the associated row. But then the rank conditions (2.4)–(2.6) applied to $\hat{A}$ would generate a family $(\hat{\sigma}_1,\ldots,\hat{\sigma}_r)$ different from $(\sigma_1,\ldots,\sigma_r)$. But $(\sigma_1,\ldots,\sigma_r)$ is an orbit invariant; hence such an $\hat{A}$ cannot exist. Equation (2.9) is proved completely analogously. ■

Let $x \in \mathbb{C}^k$ and

$$s(x) := \begin{cases} \min\{i, x_i \neq 0\} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{2.10}$$

$$(x)_s := \begin{cases} x_{s(x)} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases} \tag{2.11}$$

Exchanging $\underset{c}{\prec}$ and $\underset{r}{\prec}$ and replacing $(x)_l$ by $(x)_s$ in (2.8) and (2.9), the resulting minimal elements

$$_{R}A = \min_{\underset{c}{\prec}}\{\hat{A}\in \mathfrak{R}_A : (\hat{a}_i)_s = 1 \quad \text{for} \quad i \in \mathbf{n}\}, \tag{2.12}$$

$$_{CR}A = \min_{\underset{r}{\prec}}\{\hat{A}\in \mathcal{C}\mathfrak{R}_A : (\hat{a}^i)_s = 1 \quad \text{for} \quad i \in \mathbf{m}\} \tag{2.13}$$

again exist and are uniquely determined. The explicit characterizations of these *second echelon forms* are

$$_{R}A = [A(\hat{\sigma}_1), A(\hat{\sigma}_2),\ldots,A(\hat{\sigma}_r)] \tag{2.14}$$

with $\hat{\sigma}_i = \sigma_i$ for $i = 1,\dots,r$ and

$$A(\hat{\sigma}_i) = \begin{array}{c} \overbrace{}^{\hat{\sigma}_i} \\ \begin{bmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \\ 1 & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix}}\right\} i \\ \vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}} \end{array} \qquad \text{for} \quad i > 1,$$

$$A(\hat{\sigma}_1) = \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & 0 & & 0 \end{bmatrix}$$

and

$$_{CR}A = \left[\begin{array}{ccc|c} & 0_{n-r,r} & & 0_{n-r,m-r} \\ \hline 1 & & & \\ & \ddots & & 0_{r,m-r} \\ & & 1 & \end{array}\right]. \tag{2.15}$$

3. CONTROL CANONICAL FORMS FOR REACHABLE (A, B)-PAIRS

In order to obtain canonical forms for the similarity orbits of (A, B)-pairs, we have to consider the actions

$$A \mapsto HAH^{-1} \quad \text{and} \quad B \mapsto HB$$

simultaneously. Replacing the pairs $(A, B) \in \Sigma_{nm}^r$ by the *reachability matrix*

$$\mathfrak{R}(A, B) := \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}, \tag{3.1}$$

we can avoid this difficulty in the reachable case.

The similarity action induces row operations on $\mathcal{R}(A, B)$:

$$\mathcal{R}(HAH^{-1}, HB) = H\mathcal{R}(A, B) \qquad \forall H \in \mathrm{Gl}(n), \tag{3.2}$$

and because $\operatorname{rank} \mathcal{R}(A, B) = n$, we can construct basis matrices of $\mathbb{C}^n$, selecting certain independent columns of $\mathcal{R}(A, B)$ such that the associated representations of A and B define canonical elements in the orbits $S_{(A,B)}$. Let $m, n \in N, m, n \geqslant 1, \underline{m} = \{1, \ldots, m\}$, and $\bar{n} = \{0, 1, \ldots, n\}$. A subset $\sigma \subset \bar{n} \times \underline{m}$ is called *nice* if

$$(i, j) \in \sigma \;\Rightarrow\; (i-1, j) \in \sigma \text{ or } i = 0 \qquad \text{for all } i, j. \tag{3.3}$$

A nice subset with precisely n elements is called a *nice selection.* Let $\sigma(A, B)$ denote the matrix obtained from $\mathcal{R}(A, B)$ by removing all columns $A^i b^j$ whose index (i, j) is not in σ. For reachable pairs (A, B) belonging to the subset

$$\mathcal{R}^{\sigma} := \{(A, B) \in \Sigma^r_{n,m} : \operatorname{rank} \sigma(A, B) = n\}, \tag{3.4}$$

one particular well-defined element in the orbit $S_{(A,B)}$ is obtained by the assignment

$$(A, B) \mapsto (A(\sigma), B(\sigma)) := \left(\sigma(A, B)^{-1} A \sigma(A, B), \sigma(A, B)^{-1} B\right). \tag{3.5}$$

However, this way a particular nice selection σ covers only the subset $\mathcal{R}^{\sigma} \subset \Sigma^r_{n,m}$, and additional procedures generating all nice selections covering the complete space $\Sigma^r_{n,m}$ are required (see M. Hazewinkel [4] and M. Hazewinkel and R. E. Kalman [5] for details).

In [7] a procedure is described which assigns to every pair (A, B) a unique nice selection σ generated from a *nice order* of $\bar{n} \times \underline{m}$.

DEFINITION 3.1. A relation of total order $\sqsubseteq$ on $\underline{n} \times \underline{m}$ is called *nice* if

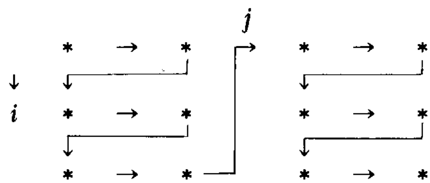
$$i \leqslant k \;\Rightarrow\; (i, j) \sqsubseteq (k, j) \quad \text{for } j \in \underline{m}, \tag{3.6}$$

$$(k, l) \sqsubseteq (i, j) \;\Rightarrow\; ((k+1), l) \sqsubseteq (i+1, j) \qquad \text{for} \quad k, i < n. \tag{3.7}$$

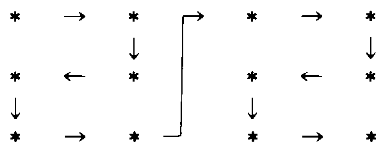
A nice order $\sqsubseteq$ can be represented graphically by a nice path through an

$(n+1)\times m$ array of points as follows.

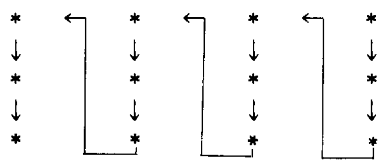
(1) Nice:



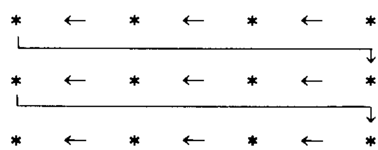
(2) Not nice:



(3) (Left) Hermite path:



(4) (Left) Kronecker path:



With every nice order $\sqsubseteq$ and pair $(A, B) \in \Sigma^r_{n,m}$ is associated the following modified Rosenbrock *deletion procedure*: Delete, in the family $(A^i b^j)_{(i,j)\in \bar{n}\times \underline{m}}$ ordered by $\sqsubseteq$, every vector $A^i b^j$ which is linearly dependent upon its predecessors. By (3.6) and the Cayley-Hamilton theorem all vectors $A^n b^j$, $j \in \underline{m}$, are deleted. By (3.7) the deletion of $A^i b^j$ implies the deletion of $A^{i+k} b^j$ for $k \geqslant 0$. Therefore the result of the deletion procedure is a nice selection $\sigma \subset \bar{n} \times \underline{m}$ with the property rank $\sigma(A, B) = n$.

We describe the canonical forms (3.5) associated to these nice orders in terms of the ($\prec$)-order. For this let $\Omega_{AB}=[w^1 \ \cdots \ w^{n+m}]$ be the $n\times(n+m)$ matrix whose columns w^k are the vectors A^ib^j, $j\in \underline{m}$ and $i\in\bar{\sigma}_j$, ordered according to $\sqsubseteq$. Further let $\tilde{\Omega}_{AB}=[\tilde{w}^1 \ \cdots \ \tilde{w}^n]$ be formed by those columns A^ib^j of Ω_{AB} for which $i\geqslant 1$. For example, if

$$\Omega_{AB}=\begin{bmatrix} b^1 & Ab^1 & b^2 & Ab^2 & b^3 & Ab^3 & A^2 & b^3\end{bmatrix}$$

then

$$\tilde{\Omega}_{AB}=\begin{bmatrix} Ab^1 & Ab^2 & Ab^3 & A^2b^3\end{bmatrix}.$$

Define the $(n+m)$-permutation matrix $\Pi(\sigma)$ by

$$\pi(\sigma)^i=\begin{cases} e^j & \text{if} \quad w^i=b^j,\\ e^{\tau_i+m} & \text{if} \quad w^i=\tilde{w}^{\tau_i}.\end{cases} \tag{3.8}$$

Proposition 3.2. *For every nice order* ($\sqsubseteq$) *on* $\bar{n}\times\underline{m}$ *and every pair* $(A,B)\in\Sigma^r_{nm}$ *we have*

$$(B(\sigma),A(\sigma))\Pi(\sigma)=\min_{\substack{\prec\\ c}}\left\{M\in\mathfrak{R}_{\Omega_{AB}}:(m_j)_s=1 \textit{ for } j\in\underline{n}\right\}. \tag{3.9}$$

Proof. $(A(\sigma),B(\sigma))$ and (A,B) are similar; hence there exists some $H\in \mathrm{Gl}(n)$ such that $\mathfrak{R}(A(\sigma),B(\sigma))=\mathfrak{R}(A,B)$. Applying $\sqsubseteq$ to $\mathfrak{R}(A(\sigma),B(\sigma))$ and $\mathfrak{R}(A,B)$, we obtain matrices Ω_{AB} and $\Omega_{A(\sigma)B(\sigma)}$, respectively, formed by the columns of $\mathfrak{R}(A,B)$ and $\mathfrak{R}(A(\sigma),B(\sigma))$ with the same column indices. Hence $\Omega_{A(\sigma_0)B(\sigma_0)}=H\Omega_{AB}$. But the columns $A(\sigma)^ib(\sigma)^j$, $i<\sigma_j$, of $\Omega_{A(\sigma)B(\sigma)}$ are $\mathbb{C}^n$ unit vectors, and the columns $A(\sigma)^{\sigma_j}b(\sigma)^j$ are linearly dependent of their predecessors in $\Omega_{A(\sigma)B(\sigma)}$. This implies [in view of the conditions (2.12) and (2.13)] that $\Omega_{A(\sigma)B(\sigma)}$ is already in second echelon form. But by construction of $\Pi(\sigma)$ we have $\Omega_{A(\sigma)B(\sigma)}=(B(\sigma),A(\sigma))\Pi(\sigma)$. From the uniqueness of the echelon forms and the characterization (2.12) we finally deduce the formula (3.9). ∎

Example 3.3. Let

$$A=\begin{bmatrix} 1 & 3 & 0 & -1 & 0\\ 0 & 5 & 1 & -4 & 0\\ 1 & 0 & -1 & -1 & -\frac{1}{2}\\ 0 & 1 & 0 & -1 & 0\\ 2 & 0 & 0 & 0 & 0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 1 & 0 & 2\\ 0 & 0 & 1\\ 4 & 0 & 4\\ 1 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}.$$

The deletion procedure associated to the *Hermite order*

$$\begin{tikzcd}
b^1 \arrow[d] & b^2 \arrow[d] & b^3 \arrow[d] \\
Ab^1 \arrow[d] & Ab^2 \arrow[d] & Ab^3 \arrow[d] \\
A^2b^1 \arrow[d] & A^2b^2 \arrow[d] & A^2b^3 \arrow[d] \\
A^3b^1 \arrow[d] & A^3b^2 \arrow[d] & A^3b^3 \arrow[d] \\
A^4b^1 \arrow[d] & A^4b^2 \arrow[d] & A^4b^3 \arrow[d] \\
A^5b^1 \arrow[uuuuur] & A^5b^2 \arrow[uuuuur] & A^5b^3
\end{tikzcd}$$

generates the Hermite list $(\sigma_1, \sigma_2, \sigma_3) = (2,3,0)$ and the basis matrix

$$\sigma(A,B) = [b^1 \quad Ab^1 \quad b^2 \quad Ab^2 \quad A^2b^2]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 4 & -4 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

The canonical form is given by

$$A(\sigma) = \sigma(A,B)^{-1} A \sigma(A,B) = \begin{bmatrix} 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix},$$

$$B(\sigma) = \sigma(A,B)^{-1} B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \\ 0 & 0 & -2 \end{bmatrix}.$$

From

$$\Omega_{AB}=\begin{bmatrix} b^1 & Ab^1 & A^2b^1 & b^2 & Ab^2 & A^2b^2 & A^3b^2 & b^3\end{bmatrix}$$
$$=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} & 2\\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -2 & 1\\ 4 & -4 & 4 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 4\\ 1 & -1 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 2\end{bmatrix}$$

we obtain

$$\Pi(\sigma)=\begin{bmatrix} e^1 & e^4 & e^5 & e^2 & e^6 & e^7 & e^8 & e^9\end{bmatrix},$$

and it is easily checked that

$$(B(\sigma),A(\sigma))\Pi(\sigma)=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} & 2\\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 2\\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -2\\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 6\\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & -2\end{bmatrix}$$

coincides with the second row echelon form of Ω_{AB}.

4. THE JORDAN CANONICAL FORM

Let $A\in\mathfrak{M}_{n,n}$, and let $(s-\lambda_i)^{n_{ij}}$, where

$$(i,j)\in I(A):=\{(i,j)\in \underline{n}\times\underline{n};\, i\in\underline{s}\wedge j\in\underline{t_i}\},\tag{4.1}$$

be the elementary divisors of A ordered according to

$$\lambda_1<\lambda_2<\cdots<\lambda_s,\tag{4.2}$$

$$n_{i1}\geqslant n_{i2}\geqslant\cdots\geqslant n_{it_i}.\tag{4.3}$$

$J(\lambda,m)$ denotes the $m\times m$ Jordan block with λ's on the diagonal, 1's on the superdiagonal, and 0's elsewhere, and

$${}^{J}A=\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{t_i}J(\lambda_i,n_{ij})\tag{4.4}$$

is the *Jordan canonical form* of A.

To interpret ${}^J\!A$ as a certain normalized minimal element of the orbit $\mathcal{S}_A$ of the similarity action

$$\begin{gathered} \mathrm{Gl}(n)\times \mathfrak{M}_{n,n} \to \mathfrak{M}_{n,n}, \\ (H,A)\mapsto HAH^{-1}, \end{gathered} \tag{4.5}$$

we adapt ${}^J\!A$ to the order $(\prec)$. For this we permute the columns and rows of ${}^J\!A$ so that the resulting matrix $J_c(A)$ has the property

$$J_c(A)=\min_{\substack{\prec\\ c}}\{P\,{}^J\!AP^T\colon P \text{ an } n\times n \text{ permutation matrix}\}. \tag{4.6}$$

EXAMPLE 4.1.

$${}^J\!A = J(\lambda_1,3)\oplus J(\lambda_1,2)\oplus J(\lambda_2,2)\oplus J(\lambda_2,2) \qquad (\lambda_1<\lambda_2)$$

$$=\left[\begin{array}{ccc|cc|cc|c} \lambda_1 & 1 & 0 & & & & & \\ 0 & \lambda_1 & 1 & & 0 & & 0 & 0 \\ 0 & 0 & \lambda_1 & & & & & \\ \hline & & & \lambda_1 & 1 & & & \\ & 0 & & 0 & \lambda_1 & & 0 & 0 \\ \hline & 0 & & & 0 & \lambda_2 & 1 & 0 \\ & & & & & 0 & \lambda_2 & \\ \hline & 0 & & & 0 & & 0 & \lambda_2 \end{array}\right],$$

$$J_c(A)=\left[\begin{array}{ccccc|ccc} \lambda_1 & 0 & 1 & 0 & 0 & & & \\ 0 & \lambda_1 & 0 & 1 & 0 & & & \\ 0 & 0 & \lambda_1 & 0 & 1 & & 0_{5\times 3} & \\ 0 & 0 & 0 & \lambda_1 & 0 & & & \\ 0 & 0 & 0 & 0 & \lambda_1 & & & \\ \hline & & & & & \lambda_2 & 0 & 1 \\ & & 0_{3\times 5} & & & 0 & \lambda_2 & 0 \\ & & & & & 0 & 0 & \lambda_2 \end{array}\right].$$

We need the following notation: For $x \in \mathbb{R}$ let

$$\rho(x) := \begin{cases} \max\{i : x_i \neq 0 \wedge i < l(x)\}, \\ 0 \quad \text{if} \quad x_i = 0 \quad \text{for } i < l(x), \end{cases} \tag{4.7}$$

$$x_\rho := \begin{cases} x_{\rho(x)} & \text{if} \quad \rho(x) \neq 0, \\ 1 & \text{if} \quad \rho(x) = 0. \end{cases} \tag{4.8}$$

For matrices $A \in M_{nm}$ let

$$\gamma(A) := \min\{t : a^t \neq 0\}, \tag{4.9}$$

$$\zeta(A) := \max\{j : a_j^{\gamma(A)} \neq 0\}, \tag{4.10}$$

$$l(A) := (\zeta(A), \gamma(A)), \tag{4.11}$$

$$(A)_l := a_{\zeta(A)}^{\gamma(A)}. \tag{4.12}$$

THEOREM 4.2.

$$J_c(A) = \min_{\prec_c} \{\hat{A} \in \mathfrak{S}_A : (\hat{a}^i)_\rho = 1 \text{ for } i \in \underline{n}\}.$$

Proof.

$$J_c(A) \in \mathfrak{N}_A := \{\hat{A} \in \mathfrak{S}_A : (\hat{a}^i)_\rho = 1 \text{ for } i \in \underline{n}\}$$

because $j_p^k \in \{0, 1\}$ for $k \neq p$. Minimal elements with respect to total orders are unique; hence it suffices to show

$$\hat{A} \succcurlyeq_c J_c(A) \qquad \forall \hat{A} \in \mathfrak{N}_A. \tag{4.13}$$

Assume that there exist elements $\hat{A} \in \mathfrak{N}_A$ with $\hat{A} \prec_c J_c(A)$. Choose $\tilde{A} = HJ_c(A)H^{-1} \in \mathfrak{N}_A$ with

$$l(\tilde{A} - J_c(A))^T = (\zeta, \gamma)^T$$
$$= \min_{\prec_c} \left\{ \begin{pmatrix} \zeta(\hat{A} - J_c(A)) \\ \gamma(\hat{A} - J_c(A)) \end{pmatrix} : \hat{A} \in \mathfrak{N}_A \wedge \hat{A} \prec_c J_c(A) \right\}. \tag{4.14}$$

Consider the columns $\tilde{a}^\gamma$, j^γ, and assume:

Case I: $\tilde{a}^\gamma_\gamma < j^\gamma_\gamma =: \lambda$. The ordering $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ of the eigenvalues of A in $J_c(A)$ implies

$$\operatorname{Ker}\left(J_c(A) - \tilde{a}^\gamma_\gamma I_n\right)^n \subset \left[e^1,\ldots,e^{\gamma-1}\right]_{\mathbf{C}}. \tag{4.15}$$

From $\tilde{a}^i = j^i$ for $i = 1,\ldots,\gamma-1$ and (4.15) we conclude

$$\left(\tilde{A} - \tilde{a}^\gamma_\gamma\right)e^\gamma \in \left[e^1,\ldots,e^{\gamma-1}\right]_{\mathbf{C}}$$

and

$$\operatorname{Ker}\left(\tilde{A} - \tilde{a}^\gamma_\gamma I_n\right)^n \supset \operatorname{Ker}\left(J_c(A) - \tilde{a}^\gamma_\gamma I_n\right)^n.$$

But

$$\dim_{\mathbf{C}} \operatorname{Ker}\left(\tilde{A} - \tilde{a}^\gamma_\gamma I_n\right)^n = \dim_{\mathbf{C}}\left(J_c(A) - \tilde{a}^\gamma_\gamma I_n\right)^n$$

then implies

$$\operatorname{Ker}\left(J_c(A) - \tilde{a}^\gamma_\gamma I_n\right)^n = \operatorname{Ker}\left(\tilde{A} - \tilde{a}^\gamma_\gamma I_n\right)^n,$$

and hence there exists a vector $z \in [e^1,\ldots,e^{\gamma-1}]_{\mathbf{C}}$ such that

$$\left(\tilde{A} - \tilde{a}^\gamma_\gamma\right)\left(e^\gamma + z\right) \in \operatorname{Ker}\left(\tilde{A} - \tilde{a}^\gamma_\gamma\right)^n,$$

and from

$$\left(e^\gamma + z\right) \in \operatorname{Ker}\left(\tilde{A} - \tilde{a}^\gamma_\gamma\right)^n \subset \left[e^1,\ldots,e^{\gamma-1}\right]_{\mathbf{C}}$$

we obtain the contradiction $e^\gamma \in [e^1,\ldots,e^{\gamma-1}]_{\mathbf{C}}$.

Case II: $\tilde{a}^\gamma_\gamma = j^\gamma_\gamma =: \lambda$ Then the column j^γ necessarily contains an off-diagonal 1-entry j^γ_q, $q < \gamma$, and we have

$$\operatorname{Ker}\left(J_c(A) - \lambda I_n\right) \subset \left[e^1,\ldots,e^{\gamma-1}\right]_{\mathbf{C}}, \tag{4.16}$$

similarly to case I. The equality $\tilde{a}^i = j^i$ for $i = 1,\dots,\gamma - 1$ now implies

$$\operatorname{Ker}(J_c(A) - \lambda I_n) = \operatorname{Ker}(\tilde{A} - \lambda I_n). \tag{4.17}$$

With (4.16) and (4.17) it is clear that $\tilde{a}^\gamma$ necessarily has nonzero entries $\tilde{a}^\gamma_j$, $j \neq \gamma$. Let

$$\tilde{A}e^\gamma = \lambda e^\gamma + e^p + \sum_{j=1}^{p-1} \tilde{a}^\gamma_j e^j, \tag{4.18}$$

and assume that $p < q$ and p is small enough that e^p intersects a Jordan block $J(j^p_p)$ with an eigenvalue $j^p_p < \lambda$. Then $(\tilde{A} - \lambda I_n)|:[e^1,\dots,e^p]_{\mathbb{C}} \xrightarrow{\sim} [e^1,\dots,e^p]_{\mathbb{C}}$ is a linear bijection, and hence there exists a vector $z \in [e^1,\dots,e^p]_{\mathbb{C}}$ such that

$$(\tilde{A} - \lambda I_n)z = e^p + \sum_{j=1}^{p-1} \tilde{a}^\gamma_j e^j.$$

With (4.18) we obtain

$$(\tilde{A} - \lambda I_n)(e^\gamma - z) = 0,$$

contradicting $\operatorname{Ker}(\tilde{A} - \lambda I_n) \subset [e^1,\dots,e^{\gamma-1}]_{\mathbb{C}}$.

If finally e^p intersects the same Jordan block $J(\lambda)$ as e^γ, we have to distinguish the following cases:

(a) the row j_p contains an off-diagonal 1-entry,
(b) $j_p = (0,\dots,0, j^p_p = \lambda, 0,\dots,0)^T$.

In *case* (a) there exists a unit vector e^r such that $(\tilde{A} - \lambda I_n)e^r = e^p$. Replacing e^γ by $\hat{e}^\gamma = \beta(e^\gamma - e^r)$ with

$$\beta = \begin{cases} 1 & \text{if} \quad \tilde{a}^\gamma_j = 0 \ \text{ for } j = 1,\dots,p-1, \\ \dfrac{1}{\tilde{a}^\gamma_\delta} & \text{if} \quad \delta := \max\{ j \in \underline{p-1} : \tilde{a}^\gamma_j \neq 0\} \text{ exists,} \end{cases}$$

we obtain

$$\tilde{A}\hat{e}^\gamma = \lambda \hat{e}^\gamma + e^\delta + \sum_{j=1}^{\delta-1} \frac{1}{\tilde{a}^\gamma_j} e^j \qquad \text{with} \quad \delta < p.$$

With

$$\tilde{H} := \left[e^1, \dots, e^{\gamma-1}, \hat{e}^\gamma, e^{\gamma+1}, \dots, e^n\right]$$

we obtain a new matrix $\hat{A} := \hat{H}\tilde{A}\hat{H}^{-1} \in \mathfrak{N}_A$ with the properties

$$\hat{A} \underset{c}{\prec} J_c(A), \qquad \gamma(\hat{A}) = \gamma(\tilde{A}), \qquad \zeta(\hat{A}) = \delta < \zeta(\tilde{A}) = p,$$

contradicting (4.14).

For *case* (*b*) let j_u be the first row in $J_c(A)$ following j_p with an off-diagonal 1-entry j_p^r. Exchanging e^r and e^γ in I_n, we obtain a new matrix $\hat{I}_n \in \mathrm{Gl}(n)$ and $\hat{I}_n\tilde{A}\hat{I}_n^{-1} \prec J_c(A)$ with $\gamma(\hat{I}_n\tilde{A}\hat{I}_n^{-1}) = r < \gamma(\tilde{A}) = \gamma$, contradicting again the minimality property (4.14) of $\tilde{A}$. ∎

REMARK 4.3. For $J_r(A) = J_c(A)^T = \min_{\underset{r}{\prec}} \{P^J A P^T\}$: P an $n \times n$ permutation matrix}, we analogously obtain the interpretation as normalized row minimal $\mathfrak{S}_A$-element:

$$J_r(A) = \min_{\underset{r}{\prec}}\left\{\hat{A} \in \mathfrak{S}_A : (\hat{a}_i)_\rho = 1 \text{ for } i \in \underline{n}\right\} \tag{4.19}$$

In both cases $J_c(A)$ and $J_r(A)$, the somewhat artificial normalizability conditions $(\hat{a}^i)_\rho = 1$ and $(\hat{a}_i)_\rho = 1$, respectively, are a consequence of the fact that the simultaneous column and row scalings

$$\begin{bmatrix} h_{11} & & 0 \\ & \ddots & \\ 0 & & h_{nn} \end{bmatrix} A \begin{bmatrix} \dfrac{1}{h_{11}} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{h_{nn}} \end{bmatrix}$$

leave the diagonal entries of A invariant. Hence in order to single out minimal-parameter orbit elements we have to normalize free off-diagonal parameters.

5. JORDAN CANONICAL FORMS FOR LINEAR SYSTEMS

A direct way to extend the canonical forms for single matrices to the similarity action on pairs $(A, B) \in \Sigma_{n,m}$ is to proceed as follows: Bring A into

Jordan canonical form. Consider the action

$$\mathrm{Stab}({}^J A) \times \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}$$

$$(S, B) \mapsto SB$$

of the *Jordan stabilizer* group

$$\mathrm{Stab}({}^J A) = \{H \in \mathrm{Gl}(n) : H\,{}^J A H^{-1} = {}^J A\}.$$

Apply the order ($\prec$) to the orbits $\mathcal{J}_{(A,B)}$ of this action and derive a *Jordan row echelon form.*

In an early paper Heymann [6] applied this idea to *controllable* systems He derived an algorithm for the construction of Jordan canonical pairs $({}^J A, {}^J B)$. However, his paper does not contain an explicit characterization of these canonical pairs.

Byrnes and Gauger (2) applied the same idea to scalar systems $(A, b) \in \Sigma_{n,1}$ and obtained canonical pairs $({}^J A, {}^J b)$ with ${}^J b \in \{0,1\}^n$. But their result requires a modification with respect to the number of 1-entries in ${}^J b$. We describe this modification in Lemma 5.2. In Theorem 5.1 the existence of a minimal-parameter Jordan row echelon form is established for $\Sigma_{n,m}$, *n and m arbitrary.* Moreover, in special cases the resulting Jordan row echelon form can be characterized solely in terms of the canonical matrices ${}^J B$ themselves.

Remember the notation $l(B) = (\zeta(B), \gamma(B))$ and $(B)_l = b_{\gamma(B)}^{\zeta(B)}$ for the leading index and the leading entry of a matrix $B \in \mathfrak{M}_{n,m}$. Let further $B(i, j)$ and $B(i)$ denote the submatrices of B corresponding to the Jordan blocks $J(\lambda_i, n_{ij})$ and $J(\lambda_i) = \oplus_{j=1}^{t_i} J(\lambda_i, n_{ij})$ of a Jordan canonical form ${}^J A$ in the pair $({}^J A, B)$.

THEOREM 5.1. *For a given pair $(A, B) \in \Sigma_{n,m}$ there exists exactly one pair $({}^J A, {}^J B) \in \mathcal{S}_{(A,B)}$ such that ${}^J A$ is of Jordan canonical form and:*

(a) ${}^J B$ is normalized, i.e.,

$$({}^J B(i,j))_l = 1 \qquad \forall (i,j) \in I({}^J A); \tag{5.1}$$

(b) we have

$${}^J B \underset{c}{\preccurlyeq} \hat{B} \tag{5.2}$$

for all $(\hat{A}, \hat{B}) \in \mathcal{S}_{(A,B)}$ with $\hat{A}$ in Jordan canonical form and $\hat{B}$ normalized.

Proof. Consider an element $\tilde{B}$ in $\mathcal{J}_{(A,B)}$ with

$$\omega(\tilde{B}) = \min_{\underset{c}{\prec}}\left\{\omega(\hat{B}) : \hat{B} \in \mathcal{J}_{(A,B)}\right\}. \tag{5.3}$$

Define an associated matrix $Q \in \mathrm{Gl}(n)$ by

$$\begin{aligned} Q &:= \operatorname{diag}\left(Q_{ij}\right)_{(i,j)\in I(A)}, \\ Q_{ij} &:= \operatorname{diag}\left(q_{ij}, q_{ij}, \ldots, q_{ij}\right) \in \mathbb{C}^{n \times n_{ij}}, \\ q_{ij} &:= \begin{cases} (B(i,j))_l & \text{if} \quad (i,j) \in I(A,B), \\ 1 & \text{if} \quad (i,j) \notin I(A,B). \end{cases} \end{aligned}$$

We obtain $((Q^{-1}B)(i,j))_l = 1$ for all $(i,j) \in I(A,B)$ and $Q^{-1}\,{}^J\!AQ = {}^J\!A$. It remains to show that ${}^JB := Q^{-1}B \underset{c}{\preccurlyeq} \hat{B}$ for all $\hat{B} \in \mathcal{J}_{(A,B)}$ for which the property (5.1) holds. Assume that there exists a matrix $S \in \mathrm{Stab}({}^J\!A)$ such that $\hat{B} = S\,{}^JB$ fulfills (5.1) but $\hat{B} \underset{c}{\prec} {}^JB$. In order to show that this is not possible we consider the elements $S \in \mathrm{Stab}({}^J\!A)$: Every S is of the form $S = \operatorname{diag}(S_1, \ldots, S_s)$, where the $S_i \in \mathrm{Stab}(J(\lambda_i))$ can be described by its t_i^2 subblocks. The (k,l)th subblock $S_i(k,l)$ of S_i is $n_{ik} \times n_{il}$ and of the form

$$\begin{bmatrix} w_1 & w_2 & w_3 & \cdot & \cdot & \cdot & w_{n_{il}} \\ 0 & w_1 & w_2 & \cdot & \cdot & \cdot & w_{n_{il}-1} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & & \cdot & w_2 \\ \cdot & & & & & & w_1 \\ \cdot & & & & & & 0 \\ \cdot & & & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \quad \text{if} \quad n_{ik} \geqslant n_{il}, \tag{5.4}$$

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & w_1 & w_2 & w_3 & \cdot & \cdot & \cdot & w_{n_{ik}} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & w_1 & w_2 & \cdot & \cdot & \cdot & w_{n_{ik}-1} \\ \cdot & & & & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & & & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & & & & \cdot & \cdot & w_2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & w_1 \end{bmatrix}$$

$$\text{if} \quad n_{ik} < n_{il}, \tag{5.5}$$

where the w_i's belong to $\mathbb{C}$ and can vary freely from block to block under the restriction that S has to be invertible.

Now let $l(S^JB - {}^JB) = (\zeta, \gamma)$, and let ${}^Jb_\zeta^\gamma$ belong to the submatrix ${}^JB(k,l)$, $(k,l) \in I({}^JA, B)$. Replace $S_k(l,l)$ in S by $\tilde{S}_k(l,l) = S_k(l,l) - I_{n_{kl}}$. If the new matrix $\tilde{S}$ is again in $\mathrm{Gl}(n)$, then we have automatically $\tilde{S} \in \mathrm{Stab}({}^JA)$ and $\tilde{S}^JB = S^JB - \tilde{B}$, where

$$\tilde{B}(i,j) = \begin{cases} 0 & \text{if} \quad (i,j) \neq (k,l), \\ {}^JB(k,l) & \text{if} \quad (i,j) = (k,l). \end{cases}$$

This implies $\omega(\tilde{S}^JB) \underset{c}{\prec} \omega({}^JB) = \omega(\tilde{B})$, contradicting (5.3). If on the other hand $\tilde{S} \notin \mathrm{Gl}(n)$, then we define

$$\gamma := \frac{\hat{b}_\zeta^\gamma - {}^Jb_\zeta^\gamma}{{}^Jb_\zeta^\gamma} \neq 0$$

and replace $S_k(l,j)$, $j = 1, \ldots, t_k$, in S by

$$S_k = (l,j) = \begin{cases} (1+\gamma)S_k(l,j) & \text{if} \quad j \neq l \\ (1+\gamma)S_k(l,l) - I_{n_{kl}} & \text{if} \quad j = l. \end{cases}$$

Then we have $\tilde{S} \in \mathrm{Stab}({}^JA)$ and

$$(\tilde{S}^JB)(i,j) = \begin{cases} \hat{B}(i,j) & \text{if} \quad (i,j) \neq (k,l), \\ \tilde{B}(i,j) & \text{if} \quad (i,j) = (k,l), \end{cases}$$

where $\tilde{b}(k,l)^j = {}^Jb(k,l)^j$ if $j = 1, \ldots, t-1$ and

$$\tilde{b}(k,l)_i^\gamma = \begin{cases} {}^Jb(k,l)_i^\gamma & \text{if} \quad i = 1, \ldots, \zeta - 1, \\ 0 & \text{if} \quad i = \zeta. \end{cases}$$

Hence again $\omega(\tilde{S}^JB) \underset{c}{\prec} \omega({}^JB) = \omega(\tilde{B})$, contradicting (5.3). Summarizing, we have shown that $S^JB \underset{c}{\succcurlyeq} {}^JB$ for all $S \in \mathrm{Stab}({}^JA)$ for which S^JB fulfills (5.1). ∎

For the explicit characterization of *Jordan canonical pairs* $({}^JA, {}^JB)$ we need some more notation. Let

$$\mu(A) = \{(i, \mu_j) : i \in \underline{s} \wedge j \in \underline{l_i}\}$$

and

$$\mu(A,b)=\{(i,j)\in\mu(A):b(i,j)\neq 0\}$$

be two index sets defined by

$$\mu_{i1}=1; \qquad \mu_{ik}=j_k, \quad k=1,\dots,l_i$$

where

$$\begin{aligned}
&n_{i1}=n_{i2}=\cdots=n_{ij_1-1}>n_{ij_1},\\
&n_{ij_1}=n_{ij_1+1}=\cdots=n_{ij_2-1}>n_{ij_2}\\
&\quad\vdots\\
&n_{ij_{l_i}}=n_{ij_{l_i}+1}=\cdots=n_{it_i}.
\end{aligned}$$

We further associate to every pair $(A,b)\in\Sigma_{n,1}$ the index lists (ζ_ν) and (ρ_ν), $\nu\in I(A)$, defined by

$$\zeta_{ij}:=l(b(i,j)) \quad \text{and} \quad \rho_{ij}:=n_{ij}-\zeta_{ij}.$$

LEMMA 5.2. *A pair (A,b) is in JCF if and only if A is in JCF and the following conditions hold:*

(i) $b(i,j)=0_{n_{ij}}$ *for* $(i,j)\notin\mu(A,b)$,

(ii) $b(i,j)=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\leftarrow\zeta_{ij}$ *for* $(i,j)\in\mu(A,b)$,

(iii) *For* $(i,j),(i,k)\in\mu(A,b)$

$$j<k \quad\Rightarrow\quad \zeta_{ij}>\zeta_{ik} \text{ and } \rho_{ij}>\rho_{ik}$$

(i.e., the numbers of zeros above 1 *and below* 1 *both decrease if j increases).*

Proof. Assume that $({}^{J}A, b)$ is in Jordan canonical form and consider a typical subcolumn

$$\begin{bmatrix} b(i,\mu_k) \\ b(i,\mu_k+1) \\ \vdots \\ b(i,\mu_{k+1}-1) \end{bmatrix} \neq 0$$

of b. Without loss of generality we can assume $b(i,\mu_k) \neq 0$. Let $S = (S_1, \ldots, S_s) \in \mathrm{Stab}({}^{J}A)$ with

(i) S_ν the identity matrix for $\nu \neq i$,
(ii) $S_i(j,j) = I_{n_{ij}}$ for $j \in \underline{t_i}$ and $j \neq \mu_k$,

(iii) $$S_i(\mu_k,\mu_k)b(i,\mu_k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow l(b(i,\mu_k)), \\ \\ \\ \\ \end{matrix}$$

(iv) $S_i(j,\mu_k)b(i,\mu_k) = -b(i,j)$ for $j = \mu_k + 1, \ldots, \mu_{k+1} - 1$,
(v) $S_i(\nu, j) = 0$ elsewhere.

Condition (iv) can be achieved because $S_i(j,\mu_k)$ is square for $j = \mu_k + 1, \ldots, \mu_{k+1} - 1$. We obtain

$$(Sb)(i,\mu_k + j) = 0$$

for

$$j = 1, \ldots, \mu_{k+1} - \mu_k - 1 \quad \text{and} \quad (Sb)(i,\mu_k) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Finally, if $(i,k) \in \mu(A,B) \ni (i,j)$, $k > j$, but $\zeta_{ij} < \zeta_{ik}$ or $\rho_{ij} < \rho_{ik}$, then there exists a matrix $S_i(j,k)$ of the form (5.4) or (5.5) such that

$$(S_i(j,k)b(i,k)+b(i,j))_{\zeta_{ij}}=0.$$

Thus we have shown that the conditions (i)–(iii) are necessary for $({}^J\!A, b)$ to be in Jordan canonical form.

Assume conversely that (i)–(iii) hold for $({}^J\!A, b)$. The normalizability condition (5.2) is fulfilled because every $b(i,j)$ is either 0 or a $\mathbb{C}^n$ unit vector. It remains to show that $b \underset{c}{\preccurlyeq} \hat{b}$ for all $\hat{b} \in \mathcal{J}_{(A,B)}$ with (5.2). Consider any subcolumn $(Sb)(i,\mu_k)$, $S\in \mathrm{Stab}({}^J\!A)$. Because of condition (i) of Lemma 5.2,

$$(Sb)(i,\mu_k)=\sum_{j=1}^{l_i} S_i(\mu_k,\mu_j)b(i,\mu_j).$$

Condition (iii) of Lemma 5.2 implies

$$l\big(S_i(\mu_k,\mu_j)b(i,\mu_j)\big)<\zeta_{ik} \qquad \text{for all} \quad (i,j)\in\mu(A,B) \quad \text{with } j\neq k.$$

Hence $l(Sb(i,\mu_k)) = l(S_i(\mu_k,\mu_k)b(i,\mu_k)) = \zeta_{ik}$. But $(Sb(i,\mu_k)_l = 1$; hence we obtain $(Sb)(i,\mu_k) \underset{c}{\succcurlyeq} b(i,\mu_k)$ for all $(i,k)\in\mu(A,b)$ or equivalently $(Sb) \underset{c}{\succcurlyeq} b$, for all $S\in\mathrm{Stab}({}^J\!A)$ for which Sb fulfills (5.2). ■

As an immediate consequence of Lemma 5.2 we obtain that the action $\mathrm{Stab}({}^J\!A)\times \mathfrak{M}_{n,1}\to\mathfrak{M}_{n,1}$ has *finitely many orbits*. Every orbit $\mathcal{J}_{(A,B)}$ is completely determined by the list of indices

$$(\zeta):=(\zeta_{ij})_{(i,j)\in\mu(A)}=\big(l({}^Jb(i,j))\big)_{(i,j)\in\mu(A)}. \tag{5.6}$$

The number of 1-entries in the blocks $b(i)$, $i=1,\dots,s$, is restricted by condition (iii) of Lemma 5.2. However, the following example shows that, opposite to the result of Byrnes and Gauger, there exist canonical vectors ${}^Jb$ with more than one nonzero 1-entry in the blocks $b(i)$, $i\in\underline{s}$.

EXAMPLE 5.3. Let $(A,b)\in\Sigma_{7,1}$ be in Jordan canonical form:

$${}^J\!A = J(\lambda_1,4)\oplus J(\lambda_1,2)\oplus J(\lambda_1,1),$$

$$\mu(A)=\{(1,1),(1,2),(1,3)\}.$$

(a) Assume $\mu(A,b)=\{(1,1),(1,2)\}$, i.e. $b(1,1)\neq 0\neq b(1,2)$. From condition (iii) of Lemma 5.2, $\zeta_{11}>\zeta_{12}>\zeta_{11}-2$. Hence the possible ζ-lists are $(3,2,0)$ and $(2,1,0)$. The associated canonical b-vectors are

$$(0010\ 01\ 0)^T \quad \text{and} \quad (0100\ 10\ 0)^T.$$

(b) Assume $\mu(A,b)=\{(1,1),(1,3)\}$, i.e. $b(1,1)\neq 0\neq b(1,3)$. From condition (iii) of Lemma 5.2, $\zeta_{11}>\zeta_{13}=1>\zeta_{11}-3$. The possible ζ-lists are $(3,0,1)$ and $(2,0,1)$, and the associated canonical b-vectors are

$$(0010\ 00\ 1)^T \quad \text{and} \quad (0100\ 00\ 1)^T.$$

(c) Assume $\mu(A,B)=\{(1,2),(1,3)\}$, i.e. $b(1,2)\neq b(1,3)$. From condition (iii) of Lemma 5.2, $\zeta_{12}>\zeta_{13}=1>\zeta_{12}-1$. This is not possible, because $\zeta_{12}\leqslant 2$.

(d) Assume $\mu(A,b)=(A)$, i.e. $b(1,j)\neq 0$ for $j=1,2,3$. From condition (iii) of Lemma 5.2, $\zeta_{12}>1>\zeta_{12}-1$. This is again not possible, because $\zeta_{12}\leqslant 2$.

(e) Assume $\mu(A,b)=\{(1,1)\}$, i.e. $b(1,1)\neq 0$. From condition (iii) of Lemma 5.2, $\zeta_{11}>0$. Hence the possible ζ-lists are $(1,0,0),(2,0,0),(3,0,0),(4,0,0)$, and the associated canonical b-vectors are

$$(1000\ 00\ 0)^T, \qquad (0100\ 00\ 0)^T,$$

$$(0010\ 00\ 0)^T, \qquad (0001\ 00\ 0)^T.$$

Table 1 gives a complete description of the orbit space of the action $\mathrm{Stab}({}^JA)\times\mathfrak{M}_{7,1}\to\mathfrak{M}_{7,1}$ for ${}^JA=J(\lambda_1,4)\oplus J(\lambda_1,2)\oplus J(\lambda_1,1)$.

We close this section with an extension of Lemma 5.2 to multivariable pairs (A,B) $(m>1)$ for which the Jordan canonical form ${}^JA$ is a Jordan block $J(\lambda,n)$.

Let ${}_nG$ be the group of all $n\times n$ matrices $S(\omega_1,\dots,\omega_n)$ of the form

$$\begin{bmatrix} \omega_1 & \omega_2 & \cdot & \cdot & \cdot & \omega_n \\ 0 & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \omega_2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \omega_1 \end{bmatrix}, \qquad \omega_1\neq 0.$$

TABLE 1

$\mu({}^J\!A, {}^J b)$	(ζ)	${}^J b$
$\varnothing$	$(0,0,0)$	$(0000\ 00\ 0)^T$
$\{(1,1)\}$	$(1,0,0)$	$(1000\ 00\ 0)^T$
	$(2,0,0)$	$(0100\ 00\ 0)^T$
	$(3,0,0)$	$(0010\ 00\ 0)^T$
	$(4,0,0)$	$(0001\ 00\ 0)^T$
$\{(1,2)\}$	$(0,1,0)$	$(0000\ 10\ 0)^T$
	$(0,2,0)$	$(0000\ 01\ 0)^T$
$\{(1,3)\}$	$(0,0,1)$	$(0000\ 00\ 1)^T$
$\{(1,1),(1,2)\}$	$(3,2,0)$	$(0010\ 01\ 0)^T$
	$(2,1,0)$	$(0100\ 10\ 0)^T$
$\{(1,1),(1,3)\}$	$(2,0,1)$	$(0100\ 00\ 1)^T$
	$(3,0,1)$	$(0010\ 00\ 1)^T$

Modifying the echelon forms of Section 2, we obtain canonical ${}_nG$ *row echelon matrices* of the form

$$\left[\begin{array}{cccccccccccccccccc}
0 & \cdots & 0 & * & \cdots & \multicolumn{1}{c|}{*} & 0 & * & \cdots & \multicolumn{1}{c|}{*} & 0 & * & \cdots & \multicolumn{1}{c|}{*} & 0 & * & \cdots & * \\
\cdot & & \cdot & \cdot & & \multicolumn{1}{c|}{\cdot} & 0 & \cdot & & \multicolumn{1}{c|}{\cdot} & 0 & \cdot & & \cdot & * & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot & * & \cdot & & \multicolumn{1}{c|}{\cdot} & 0 & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & * & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0 & & 0 & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0 & & 1 & * & \cdots & * & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cline{3-6}
0 & & 0 & 0 & \cdots & \multicolumn{1}{c|}{0} & * & * & \cdots & * & * & \cdot & & * & \cdot & \cdot & & \cdot \\
0 & & \cdot & & & \multicolumn{1}{c|}{0} & * & * & \cdots & * & * & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cline{7-10}
0 & & \cdot & & & 0 & 0 & 0 & \cdots & \multicolumn{1}{c|}{0} & * & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0 & & \cdot & & & \cdot & \cdot & \cdot & & \multicolumn{1}{c|}{0} & * & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0 & & \cdot & & & \cdot & \cdot & \cdot & & \multicolumn{1}{c|}{0} & * & * & \cdots & * & \cdot & \cdot & & \cdot \\
\cline{11-14}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & * & * & \cdots & *
\end{array}\right]$$

for the orbits of the action

$$ {}_nG \times \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,m}, \qquad (S, B) \mapsto SB \tag{5.7}$$

More precisely, we associate to $B \in \mathbb{C}^{n \cdot m}$ two index lists (ζ) and (γ) defined by

$$\gamma_1 = \min\{i \in \underline{n} : l(b^i) > 0\}$$

$$\gamma_j = \min\{i \in \underline{n} : l(b^j) > l(b^{j-1})\}, \quad \zeta_j = l(b^{\gamma_j}) \qquad \textit{for} \quad j > 1,$$

and obtain:

LEMMA 5.4. *Every orbit* ${}_n\mathcal{G}_B$ *of the action* (5.7) *contains exactly one element* $^*B$ *of the form* $[^*B(1), {}^*B(2), \dots, {}^*B(r)]$ *with*

$$^*B(1) = \begin{array}{c} \overbrace{\hphantom{0 \quad \cdots \quad 0}}^{\gamma_1} \\ \begin{bmatrix} 0 & \cdots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 0 \\ \cdot & & 1 \\ \cdot & & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} 0\\ \cdot\\ \cdot\\ \cdot\\ 0 \end{matrix}}\right\} \zeta_1 \\ \vphantom{\begin{matrix} 1\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0 \end{matrix}} \end{array},$$

$$^*B(i) = \begin{array}{c} \\ \zeta_i - \zeta_{i-1} \left\{\vphantom{\begin{matrix} 0\\ \cdot\\ \cdot\\ \cdot\\ 0 \end{matrix}}\right. \\ \vphantom{\begin{matrix} *\\ \cdot\\ \cdot\\ \cdot\\ *\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0 \end{matrix}} \end{array} \begin{array}{c} \overbrace{\hphantom{0 \quad * \quad \cdot \quad \cdot \quad \cdot \quad *}}^{\gamma_i - \gamma_{i-1}} \\ \begin{bmatrix} 0 & * & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & \cdot & & & & \cdot \\ * & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ * & * & \cdot & \cdot & \cdot & * \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} 0\\ \cdot\\ \cdot\\ \cdot\\ 0\\ *\\ \cdot\\ \cdot\\ \cdot\\ * \end{matrix}}\right\} \zeta_i \\ \vphantom{\begin{matrix} 0\\ \cdot\\ \cdot\\ \cdot\\ 0 \end{matrix}} \end{array}. \tag{5.8}$$

Proof. Without loss of generality we can assume $b^1 \neq 0$. Then there exists a matrix $S \in {}_nG$ such that

$$Sb^1 = \begin{pmatrix} 0,\dots,0, \overset{\xi_1}{\overset{\downarrow}{1}},0,\dots,0 \end{pmatrix}.$$

Assume $b^1,\dots,b^{\gamma_q - 1}$ is already in the form (5.8). Let $S(\omega_1,\dots,\omega_n) \in {}_nG$ with $\omega_1 = 1$, $\omega_j = 0$ for $j = 2,\dots,\zeta_{q-1}, \zeta_q + 1,\dots,n$ and $\zeta_{q-1}+1,\dots,\zeta_q$ such that

$$\begin{bmatrix} \omega_{\zeta_{q-1}+1} & \cdot & \cdot & \cdot & \cdot & \omega_{\zeta_q} \\ 0 & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & & \cdot & \cdot & \cdot \; 0 & \omega_{\zeta_{q-1}+1} \end{bmatrix} \begin{bmatrix} b^{\gamma}_{\zeta_{q-1}+1} \\ \vdots \\ b^{\gamma}_{\zeta_q} \end{bmatrix} = \begin{bmatrix} b^{\gamma}_1 \\ \vdots \\ b^{\gamma}_{\zeta_q - \zeta_{q-1}} \end{bmatrix}.$$

This is possible because $b^{\gamma}_{\zeta_q} \neq 0$. We obtain $S(b^j) = b^j$ for $j = 1,\dots,\gamma_q - 1$ and $(Sb^{\gamma_q})_i = 0$ for $i = 1,\dots,\zeta_i - \zeta_{i-1}, \zeta_i + 1,\dots,n$. Hence $b^1,\dots,b^q$ fulfills (5.8).

Uniqueness: Assume the orbit ${}_n\mathcal{G}_B$ contains two elements $\hat{B}$ and $\tilde{B} = S\hat{B}$ of the form (5.8). Then $\hat{b}^{\gamma_1} = S(\tilde{b}^{\gamma_1})$ implies $\omega_1 = 1$ and $\omega_2 = \omega_3 = \cdots = \omega_{\zeta_1} = 0$, and hence $\hat{b}^j = S(\tilde{b}^j)$ for $j = \gamma_1,\dots,\gamma_2 - 1$. But $\hat{b}^{\gamma_2}_{\zeta_2} \neq 0 \neq \tilde{b}^{\gamma_2}_{\zeta_2}$ together with $\hat{b}^{\gamma_2}_j = \tilde{b}^{\gamma_2}_j$ for $j = 1,\dots,\zeta_2 - \zeta_1$ implies $\omega_{\zeta_1+1} = \cdots = \omega_{\zeta_2} = 0$ and $\hat{b}^{\gamma_2}_j = \tilde{b}^{\gamma_2}_j$ for $j = \zeta_2 - \zeta_1 + 1,\dots,\zeta_2$. Continuing this way, we obtain $S\tilde{B} = \hat{B}$. ∎

The following corollary is immediate from the uniqueness part of the proof of Lemma 5.4.

Corollary 5.5. *A matrix $S(\omega_1,\dots,\omega_n)$ belongs to the stabilizer group of $^*B$ if and only if $\omega_1 = 1$ and $\omega_j = 0$ for $j = 2,\dots,\zeta^*$, where $\zeta^* = \max_{i \in \underline{m}} \zeta_i$.*

Because $\mathrm{Stab}(J(\lambda, n)) = {}_nG$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, and $S^*b \underset{c}{\prec} {}^*b$ for all $S \in {}_nG$, we obtain that the pairs $(J(\lambda, n), {}^*B)$ are Jordan canonical pairs in the sense of Theorem 5.1.

Example 5.6. The possible $(1,0,*)$ configurations for row echelon matrices $^*B \in \mathbb{C}^{3 \cdot 3}$ are shown in Table 2. The $\oplus$-entries denote nonvanishing parameters.

TABLE 2

$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 1 & * & 0 \\ 0 & 0 & \oplus \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \oplus \end{matrix}$	$\begin{matrix} 1 & 0 & * \\ 0 & \oplus & * \\ 0 & 0 & 0 \end{matrix}$
$\begin{matrix} 1 & 0 & 0 \\ 0 & \oplus & * \\ 0 & 0 & \oplus \end{matrix}$	$\begin{matrix} 1 & 0 & * \\ 0 & 0 & * \\ 0 & \oplus & * \end{matrix}$	$\begin{matrix} 0 & * & * \\ 1 & * & * \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & * & 0 \\ 1 & * & * \\ 0 & 0 & \oplus \end{matrix}$	$\begin{matrix} 0 & 0 & * \\ 1 & * & * \\ 0 & \oplus & * \end{matrix}$
$\begin{matrix} 0 & * & * \\ 0 & * & * \\ 1 & * & * \end{matrix}$	$\begin{matrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \oplus \end{matrix}$	$\begin{matrix} 0 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{matrix}$
$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & \oplus \end{matrix}$	$\begin{matrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 1 & * \end{matrix}$	$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$

4. JORDAN CONTROL CANONICAL FORMS FOR $\Sigma_{n,m}$

Another possibility for generating canonical forms for the similarity action on arbitrary, not necessarily reachable, pairs $(A, B) \in \Sigma_{n,m}$ is to combine the procedure applied in the reachable case with the Jordan canonical form. For this a nice basis of the *reachability subspace* $\mathcal{R} := \mathrm{span}_{\mathbb{C}}[B \;\; AB \;\; \cdots \;\; A^{n-1}B]$, $\dim \mathcal{R} =: n_r$, is extended to a complete basis of the state space $\mathbb{C}^n$ such that in the associated Kalman decomposition of (A, B)

$$n_r \left\{ \begin{bmatrix} \overbrace{A_1}^{n_r} & A_2 \\ 0 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right\} n_r . \tag{6.1}$$

(A_1, B_1) is in control canonical form, A_4 in Jordan canonical form, and A_2 somehow canonically fixed with a few parameters.

Let $(A, B) \in \Sigma_{n,m}$, $\sqsubseteq$ be a nice order on $\bar{n} \times \underline{m}$, and σ be the nice selection generated by $\sqsubseteq$ and (A, B). Let $(A_1(\sigma), B_1(\sigma))$ be the σ-canonical form for the pair (A_1, B_1) in the decomposition (4.1).

THEOREM 6.1. *For every pair* $(A,B)\in\Sigma_{n,m}$ *and nice order* $\sqsubseteq$ *there exists exactly one element* $({}^cA,{}^cB)\in\mathcal{S}_{(A,B)}$ *with the following properties*:

(a) $({}^cA,{}^cB)$ *is of the form*

$$\left(\begin{bmatrix} A_1(\sigma) & A_2 \\ 0 & A_3 \end{bmatrix},\begin{bmatrix} B_1(\sigma) \\ 0 \end{bmatrix}\right); \tag{6.2}$$

(b) *we have*

$$\rho({}^ca^i)=1 \qquad \textit{for} \quad i=n_r+1,\dots,n; \tag{6.3}$$

(c) *we have*

$$({}^cA,{}^cB)\underset{c}{\preccurlyeq}(\hat{A},\hat{B}) \tag{6.4}$$

for all $(\hat{A},\hat{B})\in\mathcal{S}_{(A,B)}$ *of the form* (6.2) *with the property* (6.3).

Proof. Let $\bar{A}$ denote the induced $\mathbb{C}$-linear map $\bar{A}:\mathbb{C}^n/\mathfrak{R}\to\mathbb{C}^n/\mathfrak{R}$, and let

$$J_c(\bar{A})=\bigoplus_{i=1}^{\bar{s}}\bigoplus_{j=1}^{\bar{t}_i}J(\bar{\lambda}_i,\bar{n}_{ij})$$

be the Jordan canonical form of $\bar{A}$. Extending the nice σ-basis of $\mathfrak{R}$ by "cyclic" representatives $(A-\bar{\lambda}_i)^{\bar{n}_{ij}-\nu}z_{ij}$, $\nu=1,\dots,\bar{n}_{ij}$, of the classes $(\bar{A}-\bar{\lambda}_i)^{\bar{n}_{ij}-\nu}[z_{ij}]$ of a $\mathbb{C}^n/\mathfrak{R}$ Jordan basis, we obtain a representation of the form

$$\begin{bmatrix} A_1(\sigma) & A_2 \\ 0 & J_{\bar{A}} \end{bmatrix} \tag{6.5}$$

for A, where the $n_r\times\bar{n}_{ij}$ submatrices $A_2(\bar{\lambda}_i,\bar{n}_{ij})$ of A_2 corresponding to the Jordan blocks $J(\bar{\lambda}_i,\bar{n}_{ij})$ are of the form

$$\left[\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix},0_{n_r\times(\bar{n}_{ij}-1)}\right] \tag{6.6}$$

The $*$'s denote the coordinates of $(A-\bar{\lambda})^{\bar{n}_{ij}} z_{ij} \in \mathfrak{R}$ with respect to the nice σ-basis of $\mathfrak{R}$. Multiplying the generators z_{ij} of the Jordan blocks $J(\bar{\lambda}_i, \bar{n}_{ij})$ with suitable constants $\beta_{ij} \in \mathbb{C}$, we can normalize the leading coordinates in the $(*)$-columns of $A_2(\bar{\lambda}_i, \bar{n}_{ij})$. Hence the resulting columns of $\begin{pmatrix} A_2 \\ J_{\bar{A}} \end{pmatrix}$ fulfill the normalizability condition (6.3).

Let now $\tilde{O}_A$ denote the set of all $\hat{A} \in \mathbb{C}^{n\times n}$ for which there exists a $\hat{B} \in \mathbb{C}^{n\times m}$ such that $(\hat{A}, \hat{B}) \in \mathbb{S}_{(A,B)}$, $\hat{A}$ is of the form (6.5), and $\hat{A}_2$ is of the form (6.6) with the leading coordinates in the $(*)$-columns normalized to 1. Define

$$\omega^* = \min_{\underset{c}{\prec}} \{ \omega(\hat{A}_2) : \hat{A} \in \tilde{O}_A \}. \tag{6.7}$$

ω^* exists and is uniquely determined. Let $\tilde{A} \in \tilde{O}_A$ with $\omega(\tilde{A}) = \omega^*$. We show

$$\tilde{A}_2 \underset{c}{\preccurlyeq} \hat{A}_2 \qquad \forall \hat{A} \in \tilde{O}_A. \tag{6.8}$$

Assume there exists a matrix $\hat{A} \in \tilde{O}_A$ with $\hat{A}_2 \underset{c}{\prec} \tilde{A}_2$. Let $\hat{A} = H^{-1}\tilde{A}H$. The first n_r columns in H coincide with the first unit vectors $e^1, \ldots, e^{n_r}$ of $\mathbb{C}^n$. This is due to the fact that the stabilizer group for the pair $(A_1(\sigma), B_1(\sigma))$ is trivial $(\{I_{n_r}\})$. Let now

$$t := \min\{ j \in \underline{n} : \hat{a}^j \neq \tilde{a}^j \},$$

$$e^t - h^t = (A-\lambda)^p (e^s - h^s),$$

and $e^{r_i} - h^{r_i} = (A-\lambda)^i (e^s - h^s)$ for $i = 0, \ldots, p$.

Assume $e^t - h^t \notin [e^i : i \in \underline{n} \wedge i \neq t]_{\mathbb{C}}$. Replacing h^{r_i}, $i = 0, \ldots, p$, in $H = (h^1, \ldots, h^n)$ by $(e^{r_i} - h^{r_i})$, we obtain a new matrix $\tilde{H} \in \mathrm{GL}(n)$. This can be shown as follows: assume there exists a linear combination $\sum_{i=1}^n \alpha_i \tilde{h}^i = 0$ and not all α_i are zero. Linear independence of $\{e^i : i \in \underline{n} \wedge i \notin \{r_0, \ldots, r_p\}\}$ implies that $r_k = \max\{ r \in \{r_0, \ldots, r_p\} : \alpha_r \neq 0 \}$ exists. But from $\tilde{h}^{r_k} = (A-\lambda)^k (e^s - h^s) = (1/\alpha_{r_k}) \sum_{i \neq r_k} \alpha_i \tilde{h}^i$ we conclude that $(A-\lambda)^p (e^s - h^s) = (e^t - h^t) \in [e^i : i \in \underline{n} \wedge i \notin \{r_0, \ldots, r_p\}]_{\mathbb{C}}$, contradicting our assumption. Finally, by construction we have $\omega(\tilde{a}^t - \hat{a}^t) \underset{c}{\prec} \omega(\tilde{a}^t)$, and multiplying $e^t - h^t$ by a suitable constant $\beta_t \in \mathbb{C}$, we obtain $\tilde{H}^{-1}\tilde{A}\tilde{H} \in \tilde{O}_A$ with $\omega((\tilde{H}^{-1}\tilde{A}\tilde{H})_2) \prec \omega^*$, contradicting (6.7).

If on the other hand $e^t - h^t \in [e^i; i \in \underline{n} \wedge i \neq t]_{\mathbb{C}}$, we proceed as follows. Define

$$\alpha := \max\{j: (\tilde{a}^t)_j \neq (\hat{a}^t)_j\}, \qquad \gamma := \frac{-(\tilde{a}^t)\alpha}{(\tilde{a}^t)_\alpha - (\hat{a}^t)_\alpha},$$

$$\tilde{h}^t := e^t + \gamma(e^t - h^t).$$

Replacing e^{r_i}, $i = 0, \ldots, p$, by $\tilde{h}^{r_i} = (A - \lambda)^i\ [e^s - \gamma(e^s - h^s)]$, we obtain as before a new element $\tilde{H}$ of $\mathrm{Gl}(n)$. Because $\tilde{a}^t + \gamma(\tilde{a}^t - \hat{a}^t) \underset{c}{\prec} \tilde{a}^t$ and $(A-\lambda)\tilde{h}^t \in \mathfrak{R}$, we have $\tilde{H}^{-1}\tilde{A}\tilde{H} \in \tilde{O}_A$ with $\omega((\tilde{H}^{-1}\tilde{A}\tilde{H})_2) \underset{c}{\prec} \omega(\tilde{A}_2) = w^*$.

Finally, let $P_\prec$ be the $(n - n_r)\times(n - n_r)$ permutation matrix such that $J_c(\bar{A}) = P_\prec {}^J\bar{A}P_\prec^T$ (see Section 4). Define

$${}^cA := \begin{bmatrix} A_1(\sigma) & A_2 P_\prec^T \\ 0 & J_c(\bar{A}) \end{bmatrix}, \qquad {}^cB := \begin{bmatrix} B_1(\sigma) \\ 0 \end{bmatrix}.$$

$({}^cA, {}^cB)$ is similar to (A, B) and fulfills conditions (a) and (b) of the theorem. It remains to show condition (c). Assume there exists a pair $(\hat{A}, \hat{B}) \in \mathcal{S}_{(A,B)}$ of the form (6.2) fulfilling (6.3) with $\hat{A} \underset{c}{\prec} {}^cA$. Then we have $(\hat{A}_4) \underset{c}{\preccurlyeq} J_c(\bar{A})$. But $\hat{A}_4$ and $J_c(\bar{A})$ are similar and $J_c(\bar{A})$, is the normalized minimal element in $\mathcal{S}_{\bar{A}}$ (see Theorem 4.2). Hence $\hat{A}_4 = J_c(\bar{A})$, and we necessarily have $\hat{A}_2 \underset{c}{\prec} {}^cA_2$. But $\hat{A}_2$ and ${}^cA_2$ are of such a form [see (6.6)] that $\hat{A}_2 \underset{c}{\prec} {}^cA_2$ implies $\hat{A}_2 P_\prec \underset{c}{\prec} {}^cA_2 P = \tilde{A}_2$. This contradicts (6.8). ∎

Pairs $(A, B) \in \Sigma_{n,m}$ fulfilling the properties (6.2)–(6.4) are called *σ-canonical*. The following corollary is an immediate consequence of the proof of Theorem 6.1.

COROLLARY 6.2. *A σ-canonical pair $(A, B) \in \Sigma_{n,m}$ has the following properties*:

(a) *A is of form*

$$\begin{bmatrix} A_1(\sigma) & \tilde{A}_2 \\ 0 & J_c(\bar{A}) \end{bmatrix}, \tag{6.9}$$

(b) $\tilde{A}_2(\bar{\lambda}_i, \bar{n}_{ij})$ *is of the form*

$$\begin{bmatrix} \begin{pmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, 0_{n_r \times \bar{n}_{ij}} \end{bmatrix}. \tag{6.10}$$

Here $\tilde{A}_2(\bar{\lambda}_i, \bar{n}_{ij})$ *denotes the* $n_r \times \bar{n}_{ij}$ *submatrix of* $\tilde{A}_2$ *corresponding to the submatrix* $\tilde{J}(\bar{\lambda}_i, \bar{n}_{ij}) = P_{\prec} J(\bar{\lambda}_i, \bar{n}_{ij}) P_{\prec}^T$ *of* $J_c(\bar{A})$.

EXAMPLE 6.3. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & -2 & 2 & \frac{1}{3} \\ 3 & -1 & 1 & 0 \\ 3 & -3 & 6 & -2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathcal{R} = \operatorname{span}_{\mathbb{C}}[b, Ab] = \operatorname{span}_{\mathbb{C}} \left[\begin{pmatrix} 1 \\ 3 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right].$$

There exists only one nice order $\sqsubseteq$ on $\bar{4} \times \underline{1}$. $H = [b \;\; Ab \;\; e^4 \;\; e^1]$ brings (A, B) into "partial" canonical form:

$$H^{-1}AH = \left[\begin{array}{cccc} 0 & 0 & \frac{1}{3} & 0 \\ 1 & -1 & \frac{1}{3} & 3 \\ \hline 0 & 0 & -2 & 3 \\ 0 & 0 & -\frac{1}{3} & 0 \end{array}\right], \qquad H^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right],$$

det $\bar{A} = (s+1)^2$. The set of admissible Jordan generators bringing $\begin{pmatrix} -2 & 3 \\ -\frac{1}{3} & 0 \end{pmatrix}$

into Jordan canonical form is

$$\mathcal{J} = \{x \in \mathbb{C}^4 : b, Ab, x, [A - (-1)I_4]x \text{ linearly independent}\}$$
$$= \{x \in \mathbb{C}^4; x_3 \neq 3x_4\}$$

Minimizing the nonzero entries in

$$(A + I_4)^2 x = (x_1 + x_4, x_1 + 2x_4 - \tfrac{1}{3}x_3, 0, 0)^T$$

with respect to $\underset{c}{\prec}$, we obtain the condition $x_1 = \frac{1}{3}x_3 - 2x_4$. The first component in $(A + I_4)^2 x$ then is $\frac{1}{3}x_3 - x_4$. But this component cannot be canceled, because $x \in J$. Let $\bar{x} \in \{x \in \mathbb{C}^4 : x_1 = -x_4 \ \wedge \ \frac{1}{3}x_3 - x_4 = 1\}$; for example $\bar{x} = (1\ 0\ 3\ 0)^T$. We obtain the desired canonical form with the similarity transformation $H = [b\ \ Ab\ \ (A + I_4)\bar{x}\ \ \bar{x}]$:

$$H^{-1}AH = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right], \qquad H^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right].$$

EXAMPLE 6.4. Let

$$A = \left[\begin{array}{ccccc} 0 & 0 & -1 & \alpha & 0 \\ 0 & -1 & 0 & \beta & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

A and B are canonical for the Kronecker order $\varkappa$ and every value of α and β. This is an immediate consequence of the fact that for this example we have $\mathcal{J} = \{x \in \mathbb{C}^6 : x_6 \neq 0\}$, $\det(sI - \bar{A})\bar{A} = (s+1)^2$, and $(\operatorname{ch} A)x = (\alpha x_6, \beta x_6, x_6, 0, 0)^T$.

REMARK 6.5. The conditions (6.9) and (6.10) of Corollary 6.2 are not sufficient for a pair (A, B) to be canonical. Not every (*) in the first columns of the $\tilde{A}_2(\lambda_i, \bar{n}_{ij})$-submatrices is a free parameter. The order $(\underset{c}{\prec})$ fixes zero entries below the leading 1-coordinates of these columns. Hence the invariants associated to the ${}^{c}A_2$ part of ${}^{c}A$ are given by a *family of monic polynomials*

$$(f^{\nu})_{\nu \in I({}^{c}A_4)}, \qquad f^{\nu} = f^{\nu}_{n_\nu - 1} s^{n_\nu - 1} + \cdots + f^{\nu}_0 \in \mathbb{C}[s],$$

$$f^{\nu}_k = {}^{c}a_2\left(\bar{\lambda}_i, \bar{n}_{ij}\right)^1_{k+1}, \quad k = 0, \ldots, n_\nu - 1.$$

The polynomials f^{ν} depend on $J_c(A|\mathfrak{R})$ or $\operatorname{Spec}(A|\mathfrak{R})$. If for example $\operatorname{Spec}(A|\mathfrak{R}) \cap \operatorname{Spec}(\bar{A}) = \varnothing$, then we have $f^{\nu} = 0$ for all $\nu \in I(\bar{A})$, because the maps $(A - \bar{\lambda}_i)^{n_{ij}}|: \mathfrak{R} \to \mathfrak{R}$ are bijective. ${}^{c}A_2 = 0$, i.e., there exists a direct-sum decomposition of the state space $\mathbb{C}^n = \mathfrak{R} \oplus \mathfrak{F}$, such that the system (A, B) is a direct sum of the reachable subsystem $(A|: \mathfrak{R} \to \mathfrak{R}, B: \mathbb{C}^m \to \mathfrak{R})$ and a free system $(A|:|\mathfrak{F} \to \mathfrak{F}, 0)$. In the general case the parameters of ${}^{c}A_2$ depend on the relation between the prefixed nice basis and the Jordan basis of $(\mathfrak{R}, A)$. An explicit characterization of the canonical pairs $({}^{c}A, {}^{c}B)$ solely in terms of the matrices itself is still missing. Some more detailed information about the number of free parameters in ${}^{c}A_2$, a second canonical form $({}_{c}A, {}_{c}B)$ closely related to $({}^{c}A, {}^{c}B)$, and the state-space decomposition associated to this form $({}_{c}A, {}_{c}B)$ are contained in [11].

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